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Target-Momentum and Nonlocal Effects in the High-Energy Optical Potential*

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A derivation of the optical potential for elastic scattering of high-energy nucleons by heavy nuclei is given under the multiple-scattering and impulse approximations. The momenta of the target nucleons and the dependence of the nucleon-nucleon scattering matrix on the sum **p** and difference **q** (momentum transfer) of final and initial relative momenta in the two-body center-of-mass system are taken into account in this derivation, and their effects are emphasized. The momenta of the target nucleons and the **p** dependence of the nucleon-nucleon scattering matrix, which are usually neglected in such derivations, introduce nonlocal terms in the optical potential. The contributions to this optical potential which arise from the one-pion exchange and phenomenological parts of the nucleon-nucleon scattering matrix are discussed. When the optical potential is used to describe scattering in Born approximation, the nonlocal terms have the effect of energy-dependent multiplicative operators. In particular, target-momentum effects lead to terms which, in Born approximation, can be readily interpreted as energy-dependent distortions of the optical potential in the direction of the momentum of the incident nucleon.

I. INTRODUCTION

THE optical potential for elastic scattering of high-energy nucleons by heavy nuclei has been derived by many authors in terms of the free nucleon-nucleon scattering matrix.[1–3] These derivations have been based on the multiple-scattering and impulse approximations. In the multiple-scattering approximation, collisions which result in excited intermediate states of the target nucleus are neglected although all multiple collisions which leave the target nucleus in its ground state are taken into account.[3] In the impulse approximation, formulated by Chew *et al.*[4] it is assumed that a collision between the incident nucleon and a target nucleon is essentially a collision between two free nucleons, the momentum of the target nucleon being determined by the nucleon momentum distribution in the target nucleus; that is, the effects of nuclear binding are neglected except insofar as they determine the momentum distribution of the target nucleons.

In addition to the multiple-scattering and impulse approximations, three other approximations are usually made in order to obtain a local optical potential whose terms are proportional to the coefficients of the nucleon-nucleon scattering matrix for forward scattering.[1–3] The derivation of this optical potential is sketched and the additional approximations are delineated in Sec. II to lay a foundation for the succeeding sections, in which the optical potential is investigated under the multiple-scattering and impulse approximations but without the additional approximations. These additional approximations involve the neglect of the momenta of the target nucleons in the matrix elements of the nucleon-nucleon transition operator; the neglect of the dependence of these matrix elements on the sum **p** of the final and initial relative momenta in the two-nucleon center-of-mass system, which leads to a local optical potential; and, finally, the neglect of the dependence of these matrix elements on momentum transfer, which leads to an optical potential whose radial dependence is characterized by the nucleon density in the target nucleus.[5]

The optical potential in momentum space is derived in Sec. III in terms of the form factor of the target nucleus and related functions, and in terms of the nucleon-nucleon scattering matrix in the two-body center-of-mass system and its derivatives with respect to the momentum sum **p**. In Sec. IV the optical potential in coordinate space is discussed. The contributions to the optical potential which arise from the one-pion exchange and phenomenological nucleon-nucleon scattering amplitudes are investigated with assumptions about the forms of these amplitudes for scattering off the energy shell. Results of this study are discussed in the final Sec. V.

II. DERIVATION OF THE LOCAL OPTICAL POTENTIAL

The optical potential in momentum space for elastic scattering of high-energy nucleons by heavy nuclei, derived on the basis of the multiple-scattering and im-

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[1] W. B. Riesenfeld and K. M. Watson, Phys. Rev. **102**, 1157 (1956).

[2] See H. Feshbach, Ann. Rev. Nucl. Sci. **8**, 49 (1958), for a review of derivations of the optical potential.

[3] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N. Y.) **8**, 551 (1959).

[4] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, *ibid.* **85**, 636 (1952); G. F. Chew and M. L. Goldberger, *ibid.* **87**, 778 (1952).

[5] J. Dabrowski and J. Sawicki, Nucl. Phys. **13**, 621 (1959) and J. Sawicki, *ibid.* **17**, 89 (1960) have taken some account of target-momentum and nonlocal effects in derivations of the optical potential. The optical potential in momentum space, from which they start, is equivalent to that given here in Eq. (1), but has a somewhat different form because their derivations make explicit use of correlations in the ground-state wave function of the target nucleus.

pulse approximations, can be expressed as[5,6]

$$\langle \mathbf{K}'|V|\mathbf{K}\rangle = (N/(2\pi)^{6N})\int g_0^\dagger(\mathbf{K}_\alpha',\mathbf{S}_\alpha,\mathbf{T}_\alpha) \times\exp\left[-i\sum_{\alpha=1}^{N}\mathbf{K}_\alpha'\cdot\mathbf{r}_\alpha\right] \times\langle\tfrac{1}{2}(2\mathbf{K}'-\mathbf{K}-\mathbf{K}_1)|\bar{t}|\tfrac{1}{2}(\mathbf{K}-\mathbf{K}_1)\rangle \times\exp[-i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{r}_1]g_0(\mathbf{K}_\beta,\mathbf{S}_\beta,\mathbf{T}_\beta) \times\exp\left[i\sum_{\beta=1}^{N}\mathbf{K}_\beta\cdot\mathbf{r}_\beta\right]\prod_{\alpha,\beta,\gamma=1}^{N}d\mathbf{K}_\alpha'd\mathbf{K}_\beta d\mathbf{r}_\gamma \quad (1)$$

in terms of matrix elements of the transition operator t, which describe nucleon-nucleon scattering in the two-body center-of-mass system.[2,3] The superscript bar on t indicates an average over the spin and isotopic-spin states of the target nucleons. The $\mathbf{K}$ and $\mathbf{K}_\beta$ are initial momenta of the incident and target nucleons in the coordinate system in which the target nucleus is at rest (laboratory system), $\mathbf{K}'$ and $\mathbf{K}_\alpha'$ being the corresponding final momenta. The function

$$g_0(\mathbf{K}_\beta,\mathbf{S}_\beta,\mathbf{T}_\beta)=\int \phi_0(\mathbf{r}_\beta,\mathbf{S}_\beta,\mathbf{T}_\beta) \times\exp\left[-i\sum_{\beta=1}^{N}\mathbf{K}_\beta\cdot\mathbf{r}_\beta\right]\prod_{\beta=1}^{N}d\mathbf{r}_\beta \quad (2)$$

is the N-dimensional Fourier transform of the antisymmetrized ground-state wave function ϕ_0, the variables $(\mathbf{r}_\beta,\mathbf{S}_\beta,\mathbf{T}_\beta)$ representing the space, spin and isotopic-spin coordinates of the target nucleons. The momentum-space representation (1) of the optical potential is convenient because it exhibits a separation between the two-body scattering and target-nucleus aspects of the potential.

The usual form of the local optical potential in coordinate space can be derived from (1) with the use of three additional approximations.[1–3] The first approximation $\mathfrak{A}_1$ consists of neglecting the initial momentum $\mathbf{K}_1$ of a target nucleon in the matrix elements of the two-body transition operator; that is,

$$\langle\tfrac{1}{2}(2\mathbf{K}'-\mathbf{K}-\mathbf{K}_1)|\bar{t}|\tfrac{1}{2}(\mathbf{K}-\mathbf{K}_1)\rangle \underset{\mathfrak{A}_1}{\rightarrow} \langle\tfrac{1}{2}(2\mathbf{K}'-\mathbf{K})|\bar{t}|\tfrac{1}{2}\mathbf{K}\rangle. \quad (3)$$

[6] The use of the number of target nucleons N as a factor in Eq. (1) is an approximation which is correct to relative order $1/N$. In Ref. 3, Kerman *et al.* derived an optical-potential expression corresponding to (1) in which the factor N is replaced by $(N-1)$. This optical-potential expression is correct for any N. The scattering amplitude to which it leads, however, must be multiplied by $N/(N-1)$ in order to give the correct multiple-scattering and impulse approximation to the nucleon-nucleus scattering amplitude, as was pointed out in Ref. 2 and 3. Equation (1) will, therefore, lead to the correct scattering amplitude in Born approximation and to an amplitude correct in general to relative order $1/N$, within the limitations imposed by the use of the multiple-scattering and impulse approximations.

Under approximation $\mathfrak{A}_1$, the optical potential in momentum space (1) becomes

$$\langle\mathbf{K}'|V|\mathbf{K}\rangle_{\mathfrak{A}_1}=N\langle\mathbf{k}'|\bar{t}|\mathbf{k}\rangle F(\mathbf{q}), \quad (4)$$

where

$$\mathbf{k}=\tfrac{1}{2}\mathbf{K} \quad (5a)$$

and

$$\mathbf{k}'=\tfrac{1}{2}(2\mathbf{K}'-\mathbf{K}), \quad (5b)$$

are (nonrelativistically) the initial and final relative momenta in the two-body center-of-mass system when the target nucleon is initially at rest in the laboratory, and

$$\mathbf{q}=\mathbf{K}'-\mathbf{K}=\mathbf{k}'-\mathbf{k} \quad (6)$$

is the momentum transfer. The function

$$F(\mathbf{q})=\int\rho(\mathbf{r}_1)\exp(-i\mathbf{q}\cdot\mathbf{r}_1)d\mathbf{r}_1 \quad (7)$$

is the nuclear form factor, and

$$\rho(\mathbf{r}_1)=\int\phi_0^\dagger(\mathbf{r}_\alpha,\mathbf{S}_\alpha,\mathbf{T}_\alpha)\phi_0(\mathbf{r}_\alpha,\mathbf{S}_\alpha,\mathbf{T}_\alpha)\prod_{\alpha=2}^{N}d\mathbf{r}_\alpha \quad (8)$$

is the nucleon density function (normalized to unity) of the target nucleus. The utility of approximation $\mathfrak{A}_1$ is that it leads to the form (4) in which the nuclear form factor and the nucleon-nucleon scattering matrix

$$M(\mathbf{k},\mathbf{k}')=-(m/4\pi\hbar^2)\langle\mathbf{k}'|\bar{t}|\mathbf{k}\rangle, \quad (9)$$

averaged over the spin and isotopic-spin states of the target nucleons, appear as factors; and these factors in (4) can be determined experimentally. In (9) m is the mass of a nucleon.

The optical potential in coordinate space corresponding to (4) is, with Eq. (9),

$$\langle\mathbf{r}|V|\mathbf{r}'\rangle_{\mathfrak{A}_1}=-(2\hbar^2N/(2\pi)^5m) \times\int\exp[i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]\bar{M}(\mathbf{k},\mathbf{k}')F(\mathbf{q}) \times\exp(i\mathbf{q}\cdot\mathbf{r}')d\mathbf{p}d\mathbf{q}, \quad (10)$$

where

$$\mathbf{p}=\mathbf{k}'+\mathbf{k}. \quad (11)$$

In order to obtain a local potential from (10), another approximation $\mathfrak{A}_2$ is made in which the dependence of $M(\mathbf{k},\mathbf{k}')$ on $\mathbf{p}$ is neglected; that is,

$$M(\mathbf{k},\mathbf{k}')\underset{\mathfrak{A}_2}{\rightarrow}M_E(\mathbf{q}). \quad (12)$$

In (12) $M(\mathbf{k},\mathbf{k}')$ is taken to be a function of momentum transfer only, the total energy $E=\hbar^2k^2/m$ in the two-body center-of-mass system being considered to be a constant parameter. It should be noted that Eq. (3), which characterizes approximation $\mathfrak{A}_1$, follows from the assumption in approximation $\mathfrak{A}_2$ that the scattering matrix is a function of momentum transfer only. On the other hand, approximation $\mathcal{A}_1$ does not imply approximation $\mathcal{A}_2$. With (12), the $\mathbf{p}$ integration in (10) leads to a delta function of $(\mathbf{r}'-\mathbf{r})$ so that the energy-dependent local optical potential,

$$V_{\mathcal{A}_1\mathcal{A}_2}(\mathbf{r})=-(2\hbar^2N/(2\pi)^2m)\times\int \bar{M}_E(\mathbf{q})F(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}, \quad (13)$$

follows from the general expression

$$V(\mathbf{r})\psi(\mathbf{r})=\int\langle\mathbf{r}|V|\mathbf{r}'\rangle\psi(\mathbf{r}')d\mathbf{r}'. \quad (14)$$

It should be noted that the integration in (13) is over all values of $\mathbf{q}$ so that $M_E(\mathbf{q})$ will be carried off the energy shell; that is, to values $q>2(mE)^{1/2}/\hbar$. If values of $q>2(mE)^{1/2}/\hbar$ make an appreciable contribution to (13), then $M_E(\mathbf{q})$ must be continued into this off-energy-shell region, whereas phenomenological on-energy-shell values of $M_E(\mathbf{q})$ can be used for $q\leqslant 2(mE)^{1/2}/\hbar$.[7]

The averaged scattering matrix which appears in Eqs. (10) and (13) is of the form (to order $1/N$)

$$\bar{M}=\bar{A}+\bar{C}\boldsymbol{\sigma}\cdot\hat{n}, \quad (15)$$

where $\boldsymbol{\sigma}$ is the Pauli spin operator of the incident nucleon, and $\hat{n}=\mathbf{p}\times\mathbf{q}/|\mathbf{p}\times\mathbf{q}|$ is the unit normal to the scattering plane. The form of (15) results from averaging the scattering matrix over the spin states of the target nucleons; the superscript bars on $\bar{A}$ and $\bar{C}$ indicate averages over the isotopic-spin states of the target nucleons.[8] With (15) and the assumption that the target nucleon distribution $\rho(r)$ is spherically symmetric, Eq. (13) leads to an optical potential of the form[1–3]

$$V(\mathbf{r})=V_C(r)+r^{-1}[dV_{SO}(r)/dr]\boldsymbol{\sigma}\cdot\mathbf{L}, \quad (16a)$$

with

$$V_C(r)=-(2\hbar^2N/(2\pi)^2m)\times\int[\bar{A}_E(q)]F(q)\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q} \quad (16b)$$

and[9]

$$V_{SO}(r)=(2i\hbar^2N/(2\pi)^2m)\times\int[\bar{C}_E(q)/|\mathbf{p}\times\mathbf{q}|]F(q)\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}. \quad (16c)$$

A third approximation $\mathcal{A}_3$ is frequently made in which the scattering-matrix factors which appear in square brackets in (16b) and (16c) are evaluated at $q=0$ and pulled out of the integrals; that is,

$$\bar{A}_E(q)\underset{\mathcal{A}_3}{\rightarrow}\bar{A}_E(0), \quad (17a)$$

$$\bar{C}_E(q)/|\mathbf{p}\times\mathbf{q}|\underset{\mathcal{A}_3}{\rightarrow}[\bar{C}_E(q)/|\mathbf{p}\times\mathbf{q}|]_{q=0}. \quad (17b)$$

With (17), the integrals in Eqs. (16b, c) are proportional to the nucleon density in the target nucleus [see Eq. (7)].[1–3] The justifications which have been given for the use of approximation $\mathcal{A}_3$ are that high-energy elastic scattering, which takes place primarily in the forward direction,[10] is the result of multiple two-body scatters in the forward direction; and also that, for a heavy target nucleus, the form factor $F(q)$ is a rapidly varying function of q peaked in the neighborhood of $q=0$ so that only small values of q make appreciable contributions to the integrals in Eqs. (16 b,c).[3] With Eqs. (16) and (17), the optical potential (13) becomes

$$V_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(\mathbf{r})=-(4\pi\hbar^2N/m)\{\bar{A}_E(0)\rho(r)-[\bar{C}_E(q)/|\mathbf{p}\times\mathbf{q}|]_{q=0}(1/r)(d\rho(r)/dr)\boldsymbol{\sigma}\cdot\mathbf{L}\}. \quad (18)$$

III. THE OPTICAL POTENTIAL IN MOMENTUM SPACE

In order to take account of the momenta of the target nucleons in a derivation of the optical potential, we express the matrix elements of the nucleon-nucleon transition operator, which appear in Eq. (1), in terms of the matrix elements $\langle\mathbf{k}'|\bar{t}|\mathbf{k}\rangle$ which replace them under approximation $\mathcal{A}_1$. When the former matrix elements are expressed in terms of the momenta $\mathbf{k}$ and $\mathbf{k}'$ defined in (5), they can be written

$$\langle\mathbf{k}'-\tfrac{1}{2}\mathbf{K}_1|\bar{t}|\mathbf{k}-\tfrac{1}{2}\mathbf{K}_1\rangle=\int\exp[i\mathbf{K}_1\cdot\tfrac{1}{2}(\mathbf{r}'-\mathbf{r})]\times\exp(-i\mathbf{k}'\cdot\mathbf{r}')\langle\mathbf{r}'|\bar{t}|\mathbf{r}\rangle\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}'d\mathbf{r}=\exp(-\mathbf{K}_1\cdot\nabla_p)\langle\mathbf{k}'|\bar{t}|\mathbf{k}\rangle \quad (19$$

with the help of the relation

$$\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'=-\mathbf{p}\cdot\tfrac{1}{2}(\mathbf{r}'-\mathbf{r})-\mathbf{q}\cdot\tfrac{1}{2}(\mathbf{r}'+\mathbf{r}), \quad (20)$$

which follows from (6) and (11), and the operator relation

$$\exp(i\mathbf{K}_1\cdot\mathbf{x})\exp(-i\mathbf{p}\cdot\mathbf{x})=\exp(-\mathbf{K}_1\cdot\nabla_p)\exp(-i\mathbf{p}\cdot\mathbf{x}). \quad (21)$$

Equation (19) illustrates the connection between approximations $\mathcal{A}_1$ and $\mathcal{A}_2$ mentioned following Eq. (12). If the matrix element $\langle\mathbf{k}'|\bar{t}|\mathbf{k}\rangle$ is independent of $\mathbf{p}$, the operator $\exp(-\mathbf{K}_1\cdot\nabla_p)$ can be replaced by unity; Eq. (19) then embodies both approximations $\mathcal{A}_1$ and $\mathcal{A}_2$. On the other hand, if the momentum $\mathbf{K}_1$ of

[7] R. E. Schenter and B. W. Downs, Phys. Rev. **129**, 2292 (1963).

[8] The general form of the nucleon-nucleon scattering matrix and expressions for the average coefficients $\bar{A}$ and $\bar{C}$ are given in many places; see, for example, Ref. 7 and other references cited there.

[9] Equation (16c) appears in Ref. 7 as Eq. (7b) in which the factor $(k^2\sin\theta)^{-1}$ should be replaced by $(2k^2\sin\theta)^{-1}$. The over-all factor $(N-1)$, discussed in footnote 6, was used in Ref. 7.

[10] H. A. Bethe, Ann. Phys. (N. Y.) **3**, 190 (1958).

a target nucleon is set equal to zero, Eq. (19) embodies $\mathcal{A}_1$, but not necessarily $\mathcal{A}_2$.

When (19) is inserted into Eq. (1), a transformation similar to that given in (21) can be made to replace $\exp(-\mathbf{K}_1\cdot\nabla_p)$ by $\exp(i\nabla_{r_1}\cdot\nabla_p)$. After this replacement and after the integrations over $\mathbf{k}_\alpha'$ and $\mathbf{k}_\beta$ have been carried out, Eq. (1) becomes, with Eq. (9),

$$\langle\mathbf{K}'|V|\mathbf{K}\rangle=-(4\pi\hbar^2N/m)\int\phi_0^\dagger(\mathbf{r}_\alpha,\mathbf{S}_\alpha,\mathbf{T}_\alpha)\times\exp(-i\mathbf{q}\cdot\mathbf{r}_1)\exp(i\nabla_{r_1}\cdot\nabla_p)\times\phi_0(\mathbf{r}_\alpha,\mathbf{S}_\alpha,\mathbf{T}_\alpha)\prod_{\alpha=1}^{N}d\mathbf{r}_\alpha\bar{M}(\mathbf{k},\mathbf{k}'),\quad(22)$$

where the operator $(\nabla_{r_1}\cdot\nabla_p)$ is defined by

$$(\nabla_{r_1}\cdot\nabla_p)f(\mathbf{r}_1)g(\mathbf{p})=[\nabla_{r_1}f(\mathbf{r}_1)]\cdot[\nabla_p g(\mathbf{p})].\quad(23)$$

Under the multiple-scattering and impulse approximations, Eq. (22) is an exact formal expression for the optical potential in momentum space[6] in terms of the elements of the nucleon-nucleon scattering matrix in the two-body center-of-mass system.

Evaluation of (22) can be effected with an expansion of the exponential operator. The first three terms in this expansion lead to the following expression, in which terms of relative order $1/N$ have been omitted:

$$\begin{aligned}\langle\mathbf{K}'|V|\mathbf{K}\rangle&=-(4\pi\hbar^2N/m)\{F(\mathbf{q})\bar{M}+\mathbf{F}_{(1)}(\mathbf{q})\cdot\nabla_p\bar{M}\\&+\tfrac{1}{4}[F_{(2)}(\mathbf{q})-F_{(q,q)}(\mathbf{q})]\nabla_p^2\bar{M}\\&-\tfrac{1}{4}[F_{(2)}(\mathbf{q})-3F_{(q,q)}(\mathbf{q})](\partial^2\bar{M}/\partial p_q^2)+\cdots\},\end{aligned}\quad(24)$$

where p_q is the component of $\mathbf{p}$ in the direction of $\mathbf{q}$. The form factor $F(\mathbf{q})$ is defined in Eq. (7), and

$$\mathbf{F}_{(1)}(\mathbf{q})=i\int\phi_0^\dagger\exp(-i\mathbf{q}\cdot\mathbf{r}_1)[\nabla_{r_1}\phi_0]\prod_{\gamma=1}^{N}d\mathbf{r}_\gamma,\quad(25a)$$

$$F_{(2)}(\mathbf{q})=-\int\phi_0^\dagger\exp(-i\mathbf{q}\cdot\mathbf{r}_1)[\nabla_{r_1}^2\phi_0]\prod_{\gamma=1}^{N}d\mathbf{r}_\gamma,\quad(25b)$$

$$F_{(q,q)}(\mathbf{q})=-\int\phi_0^\dagger\exp(-i\mathbf{q}\cdot\mathbf{r}_1)\times[\partial^2\phi_0/\partial r_{1,q}^2]\prod_{\gamma=1}^{N}d\mathbf{r}_\gamma;\quad(25c)$$

and $r_{1,q}$ is the component of $\mathbf{r}_1$ in the direction of $\mathbf{q}$. The functions $F_{(\ )}(\mathbf{q})$ given in (25), as well as other functions which appear when the expansion (24) is extended, are related to the nuclear form factor $F(\mathbf{q})$; for example,

$$\mathbf{F}_{(1)}(\mathbf{q})-\mathbf{F}_{(1)}^*(-\mathbf{q})=-\mathbf{q}F(\mathbf{q}),\quad(26a)$$

and

$$F_{(2)}(\mathbf{q})-F_{(2)}^*(\mathbf{q})=q^2F(\mathbf{q})-2\mathbf{q}\cdot\mathbf{F}_{(1)}^*(-\mathbf{q}).\quad(26b)$$

These functions are also related to expectation values of the momentum of a target nucleon; for example,

$$F_{(2)}(q=0)=\langle K_1^2\rangle.\quad(27)$$

The first two terms in (24) follow exactly from the first two terms in the expansion of (22), the first term giving the optical potential in momentum space under approximation $\mathcal{A}_1$, as was mentioned following Eq. (21). The third and fourth terms in (24) arise from the third term in the expansion of (22). The latter also leads to terms involving functions similar to (25c) with mixed second derivatives of ϕ_0 with respect to $r_{1,q}$ and an orthogonal coordinate or with respect to two different coordinates, both orthogonal to $r_{1,q}$. These mixed terms constitute additional contributions to (24) of order $1/N$ times those of (25b, c).[11]

The optical potential in momentum space (24) is proportional to the nucleon-nucleus scattering amplitude in Born approximation[6] and can, in principle, be used directly to describe high-energy elastic scattering in this approximation. Equation (24) is formally rather simple. When applied to real scattering (on the nucleon-nucleus energy shell), the first term is the product of two measureable quantities. In order to evaluate the two-body scattering parts of the remaining terms, however, the nucleon-nucleon scattering matrix must be specified off the energy shell even for nucleon-nucleus scattering on the energy shell. One possible simplification might be mentioned at this point. If the nucleon-nucleon scattering matrix depends upon $\mathbf{p}$ only through p^2, the second term in (24) contributes only in relative order $1/N$. This follows from the fact that $\nabla_pM(p^2)$ is in the direction of $\mathbf{p}$ and[12]

$$\mathbf{F}_{(1)}(\mathbf{q})=-\tfrac{1}{2}\mathbf{q}F(\mathbf{q})+O(1/N),\quad(28)$$

while $\mathbf{p}\cdot\mathbf{q}=0$ for scattering on the energy shell.

IV. THE OPTICAL POTENTIAL IN COORDINATE SPACE

The optical potential in coordinate space

$$\langle\mathbf{r}|V|\mathbf{r}'\rangle=(2\pi)^{-6}\int\exp[i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]\times\langle\mathbf{K}'|V|\mathbf{K}\rangle\exp(i\mathbf{q}\cdot\mathbf{r}')d\mathbf{p}d\mathbf{q}\quad(29)$$

can be obtained from the formal expression (22) for the optical potential in momentum space or from an expansion of (22) such as that given in (24). There is no advantage in using the latter because the relatively simple forms of the third and fourth terms in (24) were

[11] When the nuclear wave function ϕ_0 is taken to be a Slater determinant of single-particle shell-model functions appropriate to LS coupling, contributions to these mixed terms arise only from states with magnetic quantum numbers $m_l\neq0$ which are not paired (that is, states with $m_l\neq0$ for which there are not corresponding states with $-m_l$).

[12] The function $\mathbf{F}_{(1)}(\mathbf{q})$ contains additional terms which arise from shell-model states with unpaired magnetic quantum numbers; the remarks in footnote 11 apply here.

obtained by establishing $\mathbf{q}$ as a preferred axis; and (29) involves an integration over $\mathbf{q}$.

Each term in an expansion of (29) with (22) will involve the average nucleon-nucleon scattering matrix $\bar{M}$ or one of its derivatives. Recent analyses of nucleon-nucleon scattering data have led to a specification of a semiphenomenological nucleon-nucleon scattering matrix in which one-pion-exchange (OPE) effects are isolated.[13,14] That is, the nucleon-nucleon scattering matrix is written

$$M=M_{\mathrm{OPE}}+M', \tag{30}$$

where M_{OPE} is calculated from pion field theory, and M' is determined phenomenologically from the scattering data after OPE effects have been subtracted out.[13,14] Since the optical potential (29) is linear in M, it can be written

$$\langle\mathbf{r}|V|\mathbf{r}'\rangle=\langle\mathbf{r}|V_{\mathrm{OPE}}|\mathbf{r}'\rangle+\langle\mathbf{r}|V'|\mathbf{r}'\rangle, \tag{31}$$

corresponding to the decomposition (30). We shall consider these two contributions to the optical potential separately.

The OPE contribution to the nucleon-nucleon scattering matrix has been calculated by several authors for scattering on the energy shell in the center-of-mass system.[15–17] This OPE scattering matrix does not contain the spin-orbit coefficient C,[8] so the OPE scattering matrix, when averaged over spin and isotopic-spin states of the target nucleons, contains (to order $1/N$) only the spin-independent coefficient $\bar{A}$ [see Eq. (15)]:

$$\bar{M}_{\mathrm{OPE}}=-(g^2/4\pi\hbar c)[\hbar c/4E(k)](\tfrac{1}{4})\times[3\pm(N_N-N_P)/N][p^2/(p^2+\mu^2)]. \tag{32}$$

In (32) the plus (minus) sign on the neutron-excess term refers to an incident proton (neutron); g is the usual pseudoscalar-pseudoscalar pion-nucleon coupling constant ($g^2/4\pi\hbar c\approx 14$); μ^{-1} is the pion Compton wavelength; and $E(k)$ is the total energy of a nucleon in the center-of-mass system. In the nonrelativistic limit, the functional form of the OPE scattering matrix for scattering off the energy shell is the same as that for scattering on the energy shell.[18] We therefore take (32), with $E(k)$ replaced by mc^2, to be the general averaged scattering matrix for use in the present calculations of the optical potential.

The averaged OPE scattering matrix, which we have assumed, is a function of p alone; the $\mathbf{q}$ integration in (29) can therefore be done at once in terms of (22). The first two terms in the expansion of the resulting expression are

$$\langle\mathbf{r}|V_{\mathrm{OPE}}|\mathbf{r}'\rangle=(-\hbar^2N\bar{a}_{\mathrm{NR}}/2\pi^2m)\Big\{\rho(\mathbf{r}')\int\exp[i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]\times[p^2/(p^2+\mu^2)]d\mathbf{p}+i\boldsymbol{\varrho}_{(1)}(\mathbf{r}')\cdot\int\exp[i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]\times\nabla_p[p^2/(p^2+\mu^2)]d\mathbf{p}+\cdots\Big\}, \tag{33}$$

where $\bar{a}_{\mathrm{NR}}$ is the nonrelativistic limit of the coefficient of $[p^2/(p^2+\mu^2)]$ in (32). The target nucleon density $\rho(\mathbf{r}')$ was defined in Eq. (8), and

$$i\boldsymbol{\varrho}_{(1)}(\mathbf{r}')=i\int\phi_0^\dagger(\mathbf{r}',\mathbf{r}_2,\cdots,\mathbf{r}_N,\mathbf{S}_\alpha,\mathbf{T}_\alpha)\times\nabla_{r'}\phi_0(\mathbf{r}',\mathbf{r}_2,\cdots,\mathbf{r}_N,\mathbf{S}_\alpha,\mathbf{T}_\alpha)\prod_{\alpha=2}^{N}d\mathbf{r}_\alpha \tag{34}$$

is the Fourier transform of the "form factor" $\mathbf{F}_{(1)}(\mathbf{q})$ given in (25a). With (28), the "density" function in (34) can be written

$$\boldsymbol{\varrho}_{(1)}(\mathbf{r}')=\tfrac{1}{2}\nabla\rho(\mathbf{r}')+O(1/N). \tag{35}$$

The additional terms in the expansion (33) will contain "density" functions with higher derivatives of ϕ_0 than that in (34), coupled to higher derivatives of M_{OPE} with respect to $\mathbf{p}$.

After the integrals in (33) have been evaluated, the resulting OPE optical potential in coordinate space leads, through Eq. (14), to

$$V_{\mathrm{OPE}}(\mathbf{r})\psi(\mathbf{r})=(-\hbar^2N\bar{a}_{\mathrm{NR}}/2\pi^2m)\Big\{(2\pi)^3\rho(\mathbf{r})\psi(\mathbf{r})-2\pi^2\mu^2\int[\exp(-\mu|\mathbf{r}'-\mathbf{r}|)/|\mathbf{r}'-\mathbf{r}|]\rho(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'+2\pi^2\mu^2\int[\exp(-\mu|\mathbf{r}'-\mathbf{r}|)/|\mathbf{r}'-\mathbf{r}|]\times(\mathbf{r}'-\mathbf{r})\cdot\boldsymbol{\varrho}_{(1)}(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'+\cdots\Big\}, \tag{36}$$

in which only the first term is local. An indication of the effect of the nonlocal terms can be obtained by con-

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[13] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).

[14] See M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nucl. Sci. **10**, 291 (1960) for a review of recent analyses of nucleon-nucleon scattering data.

[15] A. F. Grashin, Zh. Eksperim. i Teor. Fiz. **36**, 1731 (1959) [translation: Soviet Phys.—JETP **9**, 1223 (1959)].

[16] J. Dabrowski, Nucl. Phys. **37**, 647 (1962).

[17] F. E. Bjorklund, B. A. Lippmann, and M. J. Moravcsik, Nucl. Phys. **29**, 582 (1962).

[18] For scattering on the energy shell in the center-of-mass system, the relation between the matrix elements of M_{OPE} and the matrix elements of the OPE contribution S_{OPE} to the nucleon-nucleon S matrix is $\langle b|S_{\mathrm{OPE}}|a\rangle=[i\hbar c/\pi E(k)]\delta^4(\ )\langle b|M_{\mathrm{OPE}}|a\rangle$, where the four-dimensional delta function expresses conservation of energy and momentum. We have assumed that the off-energy-shell matrix elements of M_{OPE} can be obtained by equating the coefficient of $\delta^4(\ )$ in $\langle b|S_{\mathrm{OPE}}|a\rangle$ to $[i\hbar c/\pi E(k)]\langle b|M_{\mathrm{OPE}}|a\rangle$. This leads to an off-energy-shell M_{OPE} which differs in form from the on-energy-shell M_{OPE} by the addition of terms of relative order $(\hbar ck/mc^2)^2$.

sidering the contribution of these terms to scattering in Born approximation. In this approximation, the radial part of the wave function is replaced by $\exp(i\mathbf{K}\cdot\mathbf{r})$, where $\mathbf{K}$ is the momentum of the incident nucleon in the laboratory. When this replacement is made in the nonlocal terms in (36), the corresponding optical potential becomes an energy-dependent multiplicative operator. If, in addition, $\rho(\mathbf{r}')$ and $\boldsymbol{\varrho}_{(1)}(\mathbf{r}')$ are expanded in Taylor's series about $\mathbf{r}'=\mathbf{r}$ and use is made of Eq. (35), the leading terms in the OPE optical potential for scattering in Born approximation are (to order $1/N$)

$$[V_{\rm OPE}(\mathbf{r})]_{\rm BA}=(-4\pi\hbar^2N\bar{a}_{\rm NR}/m)\times\{\rho(\mathbf{r})[1-\mu^2/(K^2+\mu^2)]-i\nabla_{\mathbf{r}}\rho(\mathbf{r})\cdot\mathbf{K}\mu^2/(K^2+\mu^2)^2\}. \quad (37)$$

It is interesting to note that (in this approximation) the terms involving second derivatives of the nucleon density function $\rho(\mathbf{r})$, which arise from the second and third terms in (36), cancel one another. The term in (37) which is proportional to the nucleon density is the entire nonrelativistic $V_{\rm OPE}(\mathbf{r})$ which results from a calculation in which approximations $\mathfrak{A}_1$–$\mathfrak{A}_3$ are used.[16,17,19]

The phenomenological scattering matrix M' [see Eq. (30)] will contribute coefficients $\bar{A}'$ and $\bar{C}'$ to the averaged scattering matrix (15);[20] these coefficients will ultimately lead to central and spin-orbit parts of the optical potential [see Eqs. (16)]. With the phenomenological scattering matrix M', Eqs. (14) and (29) and the first two terms of Eq. (24) lead to

$$V'(\mathbf{r})\psi(\mathbf{r})=(-4\pi\hbar^2N/(2\pi)^6m)\int\exp[i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]\times\{F(\mathbf{q})\bar{M}'+\mathbf{F}_{(1)}(\mathbf{q})\cdot\nabla_{\mathbf{p}}\bar{M}'\}\times\exp(i\mathbf{q}\cdot\mathbf{r}')\psi(\mathbf{r}')d\mathbf{p}d\mathbf{q}d\mathbf{r}'. \quad (38)$$

In order to evaluate (38), the dependence of $\bar{M}'$ on $\mathbf{p}$ and $\mathbf{q}$ must be specified. In practice, M' is determined as a function of energy and scattering angle (or energy and q^2).[20] This is just what is wanted to evaluate Eq. (38) in Born approximation, in which case (38) leads to

$$[V'(\mathbf{r})]_{\rm BA}=(-4\pi\hbar^2N/m)\times\left\{\int F(\mathbf{q})(\bar{M}'[(\mathbf{K}+\mathbf{q}),\mathbf{q}])\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}+\int\mathbf{F}_{(1)}(\mathbf{q})\cdot[\nabla_{\mathbf{p}}\bar{M}'(\mathbf{p},\mathbf{q})]_{\mathbf{p}=\mathbf{K}+\mathbf{q}}\times\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}\right\}, \quad (39)$$

with $K^2=4mE/\hbar^2$.[21]

In order to improve upon the Born-approximation result (39) and to indicate (formally at least) the nonlocal nature of (38), we seek a reasonable expansion of $\bar{M}'(\mathbf{p},\mathbf{q})$ about its on-energy-shell value $\bar{M}'[(\mathbf{K}+\mathbf{q}),\mathbf{q}]$. We, therefore, assume that $\bar{M}'(\mathbf{p},\mathbf{q})=\bar{M}'(p^2,q^2)$ (as it is on the energy shell) and that the expansion

$$\bar{M}'(p^2,q^2)=\bar{M}'[(\mathbf{K}+\mathbf{q})^2,q^2]+[p^2-(\mathbf{K}+\mathbf{q})^2][\partial\bar{M}'(p^2,q^2)/\partial p^2]_{\mathbf{p}=\mathbf{K}+\mathbf{q}}+\cdots \quad (40)$$

gives an adequate representation of $\bar{M}'$ for scattering off the energy shell.[21] Use of only the first term of (40) in (38) is equivalent to imposing approximation $\mathfrak{A}_2$ which, as we mentioned following Eq. (12), also includes the effect of approximation $\mathfrak{A}_1$. On the other hand, if the first term in (40) with $q=0$ is used in (38), this is equivalent to invoking approximations $\mathfrak{A}_1$–$\mathfrak{A}_3$. It should be noted that the use of (40) (with any value of q) in (38) will lead to an energy-dependent optical potential $V'(\mathbf{r})$ because the use of a particular value of K is implied.

With the expansion (40) and the relation (28), the optical-potential expression (38) becomes (to order $1/N$)

$$V'(\mathbf{r})\psi(\mathbf{r})=(-4\pi\hbar^2N/m)\times\{G_E(\mathbf{r})-G_{E,(1)}(\mathbf{r})[K^2+\nabla^2]+\nabla G_{E,(1)}(\mathbf{r})\cdot[2i\mathbf{K}-\nabla]+\nabla^2G_{E,(1)}(\mathbf{r})+\cdots\}\psi(\mathbf{r}), \quad (41)$$

where

$$G_E(\mathbf{r})=(2\pi)^{-3}\int F(\mathbf{q})\bar{M}'[(\mathbf{K}+\mathbf{q})^2,q^2]\times\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}, \quad (42a)$$

and

$$G_{E,(1)}(\mathbf{r})=(2\pi)^{-3}\int F(\mathbf{q})[\partial\bar{M}'(p^2,q^2)/\partial p^2]_{\mathbf{p}=\mathbf{K}+\mathbf{q}}\times\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}. \quad (42b)$$

The optical-potential expression (41) is clearly energy-dependent; and it reduces to (39) in Born approximation.

According to Eqs. (14) and (31), the optical-potential relation $V(\mathbf{r})\psi(\mathbf{r})$ is given by the sum of (36) and (38), which contain a variety of nonlocal terms whose character has been indicated in Eqs. (37) and (41). If the momenta of the target nucleons are neglected under approximation $\mathfrak{A}_1$, only the first term in each of Eqs. (33) and (38) is retained; this eliminates some, but not all, of the nonlocal terms in the optical-potential expressions (36) and (41). If the $\mathbf{p}$ dependence of the

[19] It was pointed out in Ref. 16 that the $V_{\rm OPE}$ calculated in Ref. 16 and 17 differ in sign. Equation (37) agrees with the optical potential derived in Ref. 16.

[20] See, for example, Ref. 3.

[21] Strictly speaking, the expansion (40) should be in terms of the two variables p^2 and q^2 about values consistent with the relation $q^2+p^2=K^2$, appropriate to scattering on the energy shell. In using (40) we are tacitly assuming that only values of $q^2\leqslant K^2$ contribute to the $\mathbf{q}$ integration in (38), in accordance with the discussion following Eq. (17).

nucleon-nucleon scattering matrix is neglected under approximation $\mathfrak{A}_2$, the optical potential is entirely local (with or without approximation $\mathfrak{A}_1$) because, in this case, the p integration in Eq. (29) leads to a delta function of $(\mathbf{r}-\mathbf{r}')$.

Approximation $\mathfrak{A}_3$, in which the $\mathbf{q}$ dependence of the nucleon-nucleon scattering matrix is neglected, has nothing to do with the local-nonlocal nature of the optical potential; this $\mathbf{q}$ dependence merely determines the $\mathbf{r}$ dependence of the radial functions such as those given in (42). In a previous paper,[7] we evaluated the high-energy optical potential (13)[9] under approximations $\mathfrak{A}_1$ and $\mathfrak{A}_2$ but without invoking approximation $\mathfrak{A}_3$. In that work, a phenomenological nucleon-nucleon scattering matrix $\bar{M}$ was used, in which the OPE contribution had not been isolated. That paper, therefore, contains an evaluation of the first term in Eq. (41) with $\bar{M}'$ replaced by a wholly phenomenological $\bar{M}$. The result is an energy-dependent local optical potential of the form (16a), in which the functions $V_C(r)$ and $V_{SO}(r)$ are characterized by effective density functions which have larger mean radii and more diffuse surfaces than does the nucleon distribution of the target nucleus, which characterizes these functions under approximation $\mathfrak{A}_3$ [see Eq. (18)]. An extension of that work to include the evaluation of the radial function (42b) would lead to central and spin-orbit functions $V_{C,(1)}(r)$ and $V_{SO,(1)}(r)$ similar to those just discussed, which correspond to the radial function (42a).[22]

V. CONCLUDING REMARKS

The optical-potential relations $V(\mathbf{r})\psi(\mathbf{r})$, given in Eqs. (36) and (38), can be approximated by series expansions which involve derivatives of the wave function as well as multiplicative operators. The character of those terms in the optical-potential operator which lead to derivatives of the wave function has been indicated in Eq. (41) and in the Born-approximation relation (37).

The term of the form

$$i\mathbf{K}\cdot\nabla\rho(\mathbf{r}) \tag{43a}$$

in the Born approximation (37) to $V_{\mathrm{OPE}}(\mathbf{r})$ has an interesting consequence: It corresponds to an energy-dependent distortion of the optical potential in the direction of the momentum of the incident nucleon. This distortion will be primarily in the neighborhood of the periphery of the optical potential because the nucleon density in heavy nuclei changes appreciably only in the surface region.[23]

The optical-potential relation (41) contains the operator $\nabla G_{E,(1)}(\mathbf{r})\cdot[2i\mathbf{K}-\nabla]$, similar to (43a), which reduces to

$$i\mathbf{K}\cdot\nabla G_{E,(1)}(\mathbf{r}) \tag{43b}$$

in Born approximation. The radial function $G_{E,(1)}(\mathbf{r})$ can be expected to have a form similar to that of $G_E(\mathbf{r})$ [see Eqs. (42)]. This means $G_{E,(1)}(\mathbf{r})$ will contain central and spin-orbit terms of the form given in Eq. (18) with $\rho(r)$ replaced by effective density functions such as those discussed at the end of Sec. IV. The term (43b), therefore, also has an interpretation as an energy-dependent distortion of the surface of the optical potential.

The emphasis in this paper has been on the optical potential in coordinate space. The detail with which the development in Sec. IV was pursued can, of course, be extended to the optical potential in momentum space discussed in Sec. III.

It should be recalled that several of the expressions given in this paper are valid only to relative order $1/N$ for arbitrary heavy nuclei. This restriction has been pointed out where approximate formulas [such as (28) and (35)] have been used. The reason for reporting expressions whose validity is restricted in this manner is that these restricted expressions can be more readily interpreted than can the general expressions to which they correspond; moreover, the validity of the entire optical potential discussed here is similarly restricted by the use of Eq. (15) for the averaged nucleon-nucleon scattering matrix.[24]

The validity of the development presented here depends upon the correctness of the forms taken for the constituents of the nucleon-nucleon scattering matrix for scattering off the energy shell. We have made what we consider to be reasonable assumptions about this aspect of the scattering matrix in order to indicate the qualitative features of the corresponding optical potential.

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[22] See also the reference to J. Sawicki in footnote 5.

[23] See, for example, R. Hofstadter, Rev. Mod. Phys. **28**, 214 (1956), and Ann. Rev. Nucl. Sci. **7**, 231 (1957).

[24] The approximation implied by the use of the factor N in Eq. (1) can easily be circumvented; see footnote 6.